Appendix A : Γ-space and μ-space

- In Section F, we introduced the Γ-space for the phase space of a N-particle system.

  In 3D, we need to specify

$$\{\underbrace{x_1, y_1, z_1, p_{1x}, p_{1y}, p_{1z}}_{\text{position and momentum of particle \#1}}; \cdots, \underbrace{x_N, y_N, z_N, p_{Nx}, p_{Ny}, p_{Nz}}_{\text{position and momentum of particle \#N}}\}$$

6N quantities

Γ-space : 6N dimensional space

One point in Γ-space specifies 6N quantities

$\Rightarrow$ One point in Γ-space specifies a state of the N-particle system.

- μ-space

  - This is the phase space for one particle.

    For a particle in a 3D system, its phase space is a 6-dimensional space. The 6D axes are:

    $\{x, y, z, p_x, p_y, p_z\}$.

    When a particle takes on $\{x_1, y_1, z_1, p_{1x}, p_{1y}, p_{1z}\}$, its state is represented by a point in μ-space.

Thus, a particle's state $\Rightarrow$ a point in μ-space

A N-particle state $\Rightarrow$ states of N particles

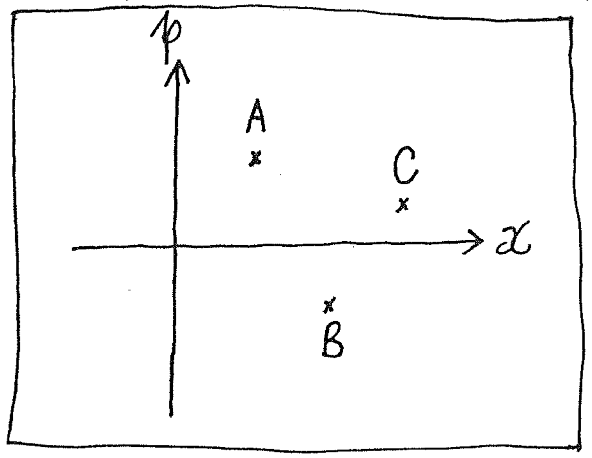
$\Rightarrow$ N points in μ-space

Ref:

Phase-space usually appears in Classical Mechanics under the Chapter on Hamilton's Dynamics. In particular, the related Liouville's theorem is often the starting point in the discussion on how a system evolves in time in Statistical Dynamics. I avoided the discussion so as not to confuse you. For applying Equilibrium Stat. Mech., we do not need that discussion.

- Example: 3 distinguishable particles in 1D

μ-space in 1D : $\{x, p\}$ (2D μ-space)



⟵ specifies the values of (x_A, p_A), (x_B, p_B), (x_C, p_C)

$\Rightarrow$ specifies one state of the 3-particle system.

- Questions:
  - how QM ideas could get into the picture?
  - how distinguishability and indistinguishability of the particles get into the picture?
  - when we are sure that there won't be two or more particles in the same "(x, p)" state, how can we relate the counting when A, B, C are distinguishable to the counting when A, B, C are indistinguishable? [Correcting over-counting] ($\frac{1}{3!}$ or $\frac{1}{N!}$ works!)

Appendix B : Stirling's Approximation

- In stat. mech., we encounter $\ln N!$ very often, where N is a hugh number (e.g. $\sim 10^{23}$)

Stirling's formula: $n! \cong \sqrt{2\pi n}\, n^n e^{-n}$ (*)

OR $\boxed{\ln N! \cong N \ln N - N + \frac{1}{2}\ln(2\pi N)} + O\left(\frac{1}{N}\right)$

e.g. $N = 50$, $\ln N! = 148.478$ (exact)

$$\left.\begin{aligned} N\ln N - N &= 145.601 \\ \tfrac{1}{2}\ln(2\pi N) &= 2.875 \end{aligned}\right\} \quad N\ln N - N \gg \tfrac{1}{2}\ln(2\pi N) \text{ for large } N$$

e.g. $N = 10^{18}$,

$N\ln N - N = 4.0446 \times 10^{19}$

$\frac{1}{2}\ln(2\pi N) \approx 21.64$ (tiny!)

$\therefore$ Practically, for typical stat. mech. situations:

$\boxed{\ln N! \approx N\ln N - N}$ Stirling's formula

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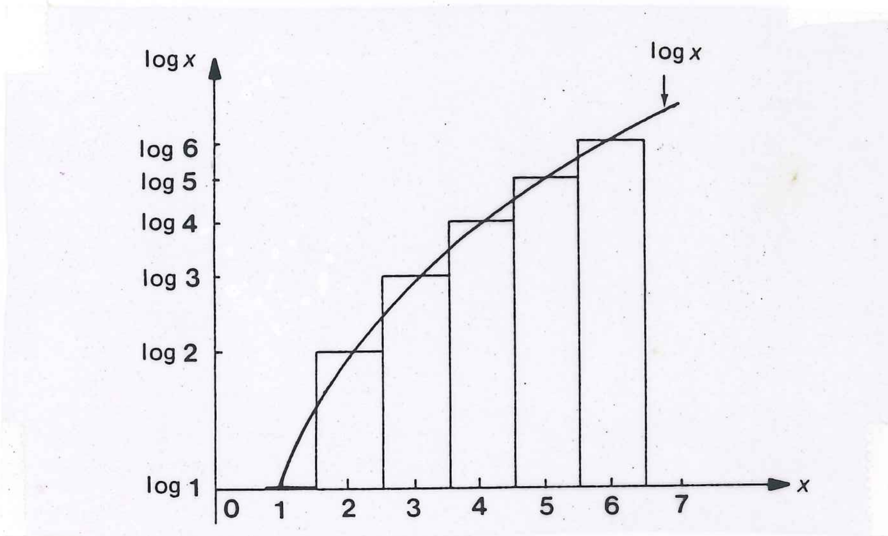
* The proof is quite involved. It is related to the Gamma function, Laplace transform, and functions of a complex variable. For a version of the proof and how the well the formula works, see Problem Set 1.

- A poorman's way of looking at the Stirling's formula.

$$\ln N! = \ln[N(N-1)(N-2)\cdots 2\cdot 1] = \sum_{i=1}^{N} \ln i$$



$\sum_{i=1}^{N} \ln i$ = sum of areas under the rectangles

- For $N \gg 1$, approximate areas under rectangular by area under the curve $\ln x$.

i.e. $\sum_{i=1}^{N} \ln i \approx \int_1^N \ln x\, dx = N\ln N - N + \underset{\ll N}{1}$

$\approx N\ln N - N$